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Linear functionals and the duality principle for harmonic functions

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Let $\mathcal{H}$ be the class of complex-valued harmonic functions in the unit disk $|z| < 1$ and $\mathcal{H}_1$ the set of all functions $f \in \mathcal{H}$ such that $f(0) = 0$, $f_z(0) = 1$ and $f_{\bar{z}}(0) = 0$. For $V \subset \mathcal{H}_1$, its dual V^* is

$$V^* = \{g \in \mathcal{H}_1 : (f * g)(z) \neq 0 \text{ for all } 0 < |z| < 1,\ f \in V\},$$

where $*$ denotes the Hadamard product for harmonic functions. The set V is a *dual class* if $V = W^*$ for some $W \subset \mathcal{H}_1$. In the present paper, the duality principle is extended to $\mathcal{H}_1$ by means of the Hadamard product. Counterparts of the dual classes are introduced and their structural properties studied.

1 Introduction and main results

Let $\mathbb{D}_R = \{z : |z| < R\}$ be an open disk in the complex plane $\mathbb{C}$ and for brevity, let $\mathbb{D} = \mathbb{D}_1$, and $\check{\mathbb{D}} = \mathbb{D}\backslash\{0\}$. Denote by $\mathcal{H}(\mathbb{D}_R)$ the class of complex-valued harmonic functions $f = u + iv$ in $\mathbb{D}_R$, where u and v are real-valued harmonic functions in $\mathbb{D}_R$. In particular, $\mathcal{H} = \mathcal{H}(\mathbb{D})$ and let $\overline{\mathcal{H}}$ be the class of harmonic functions in the closed disk $\overline{\mathbb{D}}$, that is, $\overline{\mathcal{H}} = \bigcap_{R>1} \mathcal{H}(\mathbb{D}_R)$. As in [2, p. 38], it can be shown that $\mathcal{H}(\mathbb{D}_R)$ is a metrizable topological linear space with the topology of locally uniform convergence in $\mathbb{D}_R$.

Each $f \in \mathcal{H}(\mathbb{D}_R)$ has the representation $f(z) = \phi(z) + \overline{\psi(z)}$, where ϕ and ψ are analytic in $\mathbb{D}_R$ of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n(\phi) z^n \quad \text{and} \quad \psi(z) = \sum_{n=1}^{\infty} a_n(\psi) z^n,$$

and both series being absolutely convergent for $|z| < R$. Here ϕ and ψ are respectively referred to as the analytic and co-analytic parts of f. For $f = \phi + \overline{\psi} \in \mathcal{H}(\mathbb{D}_R)$ and $g = \Phi + \overline{\Psi} \in \mathcal{H}(\mathbb{D}_R)$, the convolution (or Hadamard product) of f and g is $f * g$ defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n(\phi) a_n(\Phi) z^n + \overline{\sum_{n=1}^{\infty} a_n(\psi) a_n(\Psi) z^n}.$$

Clearly, the operation $*$ is commutative. When f and g are analytic, the definition $f * g$ coincides with the convolution of analytic functions, of which the literature is vast [8] (see also [4], [5]). In the following, for

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$f \in \mathcal{H}(\mathbb{D}_R)$, it is convenient to use the compact form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(f)p_k(z) = \left(\sum_{k=0}^{\infty} a_k(f)z^k\right) + \overline{\left(\sum_{k=1}^{\infty} \overline{a_{-k}(f)}z^k\right)},$$

where $p_k(z) = z^k$, $p_{-k}(z) = \overline{z}^k$ for $k \geq 0$. In this form, for $f(z) = \sum_{k=-\infty}^{\infty} a_k(f)p_k(z)$ and $g(z) = \sum_{k=-\infty}^{\infty} b_k(g)p_k(z)$ in $\mathcal{H}(\mathbb{D}_R)$, $f * g$ can be written in the equivalent form

$$(f * g)(z) = \sum_{k=-\infty}^{\infty} a_k(f)b_k(g)p_k(z) = \frac{1}{2\pi}\int_0^{2\pi} f\big(\rho e^{it}\big)g\big(z\rho^{-1}e^{-it}\big)\,dt, \quad |z| < \rho < R.$$

In particular, the unit element for $\mathcal{H}(\mathbb{D}_R)$, $R \leq 1$, is

$$e(z) = \sum_{k=-\infty}^{\infty} p_k(z) = \frac{1}{1-z} + \frac{\overline{z}}{1-\overline{z}} = \frac{1-|z|^2}{|1-z|^2},$$

whereas for $R > 1$ there is no such element.

Consider now the following subclasses of normalized complex-valued harmonic functions:

$$\mathcal{H}_0 = \{f \in \mathcal{H} :\, a_0(f) = 1\}, \quad \text{and}$$
$$\mathcal{H}_1 = \{f \in \mathcal{H} :\, a_0(f) = a_1(f) - 1 = a_{-1}(f) = 0\}.$$

Note that $\mathcal{H}_1$ reduces to the class of normalized analytic functions when the co-analytic part of f is identically zero in the unit disk. In [1], Clunie and Sheil-Small introduced and studied various subclasses of $\mathcal{H}_1$, particularly univalent functions (convex, starlike, close-to-convex) from $\mathcal{H}_1$. For the purpose of applications to complex-valued harmonic univalent functions, the results of this paper are stated for the class $\mathcal{H}_1$. However, all the assertions obtained remain valid (with appropriate changes) for the class $\mathcal{H}_0$.

The space Λ of all complex-valued continuous (with respect to the compact convergence in $\mathbb{D}$) linear functionals on $\mathcal{H}$ plays an important role in the statement of our results. Let $\lambda \in \Lambda$ and $f(z) = \sum_{k=-\infty}^{\infty} a_k(f)p_k(z) \in \mathcal{H}$. Then

$$\lambda(f) := \sum_{k=-\infty}^{\infty} a_k(f)\lambda(p_k) = \sum_{k=-\infty}^{\infty} a_k(f)c_k$$

and, according to Toeplitz [10], the sequence $\{c_k\}$ generates such a functional if and only if the corresponding series $g(z) = \sum_{k=-\infty}^{\infty} c_k p_k(z)$ has a radius of convergence greater than 1. In addition,

$$\lambda(f) := (g * f)(1) = \lim_{z\to 1} \frac{1}{2\pi}\int_0^{2\pi} g\big(e^{it}\big)f\big(ze^{-it}\big)\,dt.$$

More precisely, the following lemma (cf. [10]) characterizes continuous linear functionals on $\mathcal{H}$.

Lemma 1.1 *A complex-valued functional λ on $\mathcal{H}$ is continuous and linear if and only if there exists a function $g \in \overline{\mathcal{H}}$ such that $\lambda(f) = (f * g)(1)$ for all $f \in \mathcal{H}$.*

In the sequel we shall denote by $\lambda := g$ the correspondence between λ and g as described in Lemma 1.1. In fact, Lemma 1.1 can be proved by a reasoning similar to that in [2, pp. 42–43].

Ruscheweyh introduced the notions of the dual and dual hulls for classes of analytic functions. In [6], [7], a certain analogy was established between the duals and dual hulls on one side, and the adjoint and the second adjoint spaces, on the other. In some interesting cases the second dual class proves to be surprisingly large and, as a consequence, many new as well as already known facts fall into a simple pattern. In particular, Ruscheweyh's approach (see [6], [7]) is useful for numerous applications in geometric function theory and even beyond. In the present paper, this technique is extended to the case of harmonic functions, and certain properties of the relevant notions are studied under this new setting.

For $V \subset \mathcal{H}_1$, define its dual V^* as

$$V^* = \{g \in \mathcal{H}_1 : (f * g)(z) \neq 0 \text{ for all } z \in \check{\mathbb{D}}, \ f \in V\}.$$

We say that V is a *dual class* if $V = W^*$ for some $W \subset \mathcal{H}_1$. A number of basic properties of dual class can be observed. For example, it is easy to see that $V \subset \mathcal{H}_1$ is a dual class if and only if $V = V^{**}$, the latter being defined as $V^{**} = (V^*)^*$. Evidently, V^{**} is the smallest dual class containing V, so it is called the *dual hull* of V. For $V \subset \mathcal{H}_1$ and $U \subset \overline{\mathcal{H}}_1$, we define $V^\top$ and $U^\perp$ respectively by

$$V^\top = \{g \in \overline{\mathcal{H}}_1 : (f * g)(1) \neq 0 \text{ for all } f \in V\},$$

and

$$U^\perp = \{h \in \mathcal{H}_1 : (g * h)(1) \neq 0 \text{ for all } g \in U\},$$

where $\overline{\mathcal{H}}_1$ is the class of functions in $\mathcal{H}_1$ that are harmonic in the closed disk $\overline{\mathbb{D}}$. In accordance with a definition from [6], a subclass V of $\mathcal{H}_1$ is called *complete*, provided that $P_x f \in V$ for any $f \in V$ and any $x \in \overline{\mathbb{D}}$, where $(P_x f)(z) = f(xz)/x$ for $x \neq 0$ and $(P_0 f)(z) = z$. Finally, the *complete hull* of V is defined by

$$V' := \bigcup_{x \in \overline{\mathbb{D}}} P_x V = \{P_x f : f \in V, x \in \overline{\mathbb{D}}\},$$

which is the smallest of all complete sets containing V. Observe that for each $V \subset \mathcal{H}_1$ there holds $(V')^* = V^*$, and for a compact V, the set V' is also compact.

Our main results are stated in the following theorems. Their counterparts for the analytic case were proved by Nezhmetdinov [3]. However, a representation $V^* = \overline{(V')^\top}$, also established there, in general fails for the harmonic case. Indeed, if we take $V = \{e(z)\} \subset \mathcal{H}_0$, then the sequence $\{f_n\}_{n \geq 1}$ of functions defined by

$$f_n(z) = 1 + \frac{n}{n+i}(z + \overline{z})$$

lies in V^*, whereas its limit function $f(z) = 1 + z + \overline{z}$ does not. This example also shows that V^* is not closed, unlike the analytic case.

Theorem 1.2 *Let V be a compact subset of $\mathcal{H}_1$ and $V^\top$ be complete. Then $\lambda(V) = \lambda(V^{**})$ for each $\lambda \in \Lambda$. Moreover, $f \in V^{**}$ if and only if $\lambda(f) \in \lambda(V)$ for all $\lambda \in \Lambda$.*

Here $\lambda(V)$ stands for the set $\{\lambda(f) : f \in V\}$.

Theorem 1.3 *Under the assumptions of Theorem 1.2, then $V^{**} = \left(V^\top\right)^\perp$.*

Theorem 1.4 *Let $V \subset \mathcal{H}_1$ be a compact set such that $V^\top$ is complete. Then*

$$V^\top = \bigcup_{0<r<1} P_r\left(V^\top\right) = \bigcup_{0<r<1} P_r\left[(P_r V)^\top\right].$$

The proofs of Theorems 1.2–1.4, as well as their consequences, will be given in Section 3.

2 Preliminary lemmas

To begin with, several auxiliary assertions are established.

Lemma 2.1 *Suppose that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, in the spaces $\mathcal{H}(\mathbb{D}_{R_1})$ and $\mathcal{H}(\mathbb{D}_{R_2})$, respectively. Then $f_n * g_n \to f * g$, as $n \to \infty$, in $\mathcal{H}(\mathbb{D}_{R_1 R_2})$.*

Proof. Let $z \in \overline{\mathbb{D}_\rho}$, $0 < \rho < R_1 R_2$. Choose $\rho_1, \rho_2 \in (0, 1)$ such that $\rho_1 < R_1$, $\rho_2 < R_2$ and $\rho < \rho_1\rho_2 < R_1 R_2$. Since $f_n \to f$ as $n \to \infty$ in $\mathcal{H}(\mathbb{D}_{R_1})$, we deduce that this sequence converges uniformly to f in $\overline{\mathbb{D}_{\rho_1}}$, and, in particular, it is uniformly bounded on $\overline{\mathbb{D}_{\rho_1}}$. Therefore, Cauchy inequalities for the coefficients yield

$$|a_{\pm k}(f)| \leq M_1 \rho_1^{-k}, \tag{2.1}$$

$$|a_{\pm k}(f_n) - a_{\pm k}(f)| \leq \varepsilon_{1,n} \rho_1^{-k}, \quad \text{for all} \quad n \geq 1, \ k \geq 0, \tag{2.2}$$

where $M(r,f) = \sup_{|z|=r}|f(z)|$, $M_1 = M(\rho_1, f)$ is some positive constant, $\varepsilon_{1,n} = M(\rho_1, f_n - f)$ and $\varepsilon_{1,n} \to 0$, $n \to \infty$. In a similar way,

$$|a_{\pm k}(g_n)| \le M_2\rho_2^{-k} \quad \text{and} \quad |a_{\pm k}(g_n) - a_{\pm k}(g)| \le \varepsilon_{2,n}\rho_2^{-k} \quad \text{for} \quad n \ge 1, k \ge 0, \tag{2.3}$$

where $M_2 = M(\rho_2, g)$ and $\varepsilon_{2,n} = M(\rho_2, g_n - g) \to 0$ as $n \to \infty$. From the estimates (2.1)–(2.3), for all $z \in \overline{\mathbb{D}_\rho}$, it follows that

$$\begin{aligned} |(f_n * g_n)(z) - (f * g)(z)| &\le \sum_{k=-\infty}^{\infty} \big|a_k(f_n)a_k(g_n) - a_k(f)a_k(g)\big|\rho^{|k|} \\ &= \sum_{k=-\infty}^{\infty} \big|a_k(f_n - f)a_k(g_n) + a_k(f)a_k(g_n - g)\big|\rho^{|k|} \\ &\le \sum_{k=-\infty}^{\infty} \left[\frac{\varepsilon_{1,n}}{\rho_1^{|k|}}\frac{M_2}{\rho_2^{|k|}} + \frac{\varepsilon_{2,n}}{\rho_2^{|k|}}\frac{M_1}{\rho_1^{|k|}}\right]\rho^{|k|} \\ &= (\varepsilon_{1,n}M_2 + \varepsilon_{2,n}M_1)\frac{\rho_1\rho_2 + \rho}{\rho_1\rho_2 - \rho}. \end{aligned}$$

Since the last expression tends to 0 as $n \to \infty$, the proof is complete. □

Lemma 2.2 *Let $V \subset \mathcal{H}(\mathbb{D}_{R_1})$ be compact, $g \in \mathcal{H}(\mathbb{D}_{R_2})$ and $U = \{g * f : f \in V\}$. Then U is compact in $\mathcal{H}(\mathbb{D}_{R_1R_2})$.*

Proof. Note that for any sequence $\{g * f_n\}$ in U, in view of the compactness of V, a subsequence f_{n_k} can be chosen to converge to some $f \in V$ in $\mathcal{H}(\mathbb{D}_{R_1})$. From Lemma 2.1, it follows that $g * f_{n_k} \to g * f \in U$ as $k \to \infty$, which shows that U is compact. □

Lemma 2.3 *Let V be a compact set in $\mathcal{H}(\mathbb{D}_R)$, $R > 1$, such that $f(1) \neq 0$ for all $f \in V$. Then there exists a real number $\sigma \in (1, R)$ with $f(\sigma) \neq 0$ for all $f \in V$.*

Proof. Assume the contrary. Then there is a sequence $\{x_n\}_{n\ge1}$ of real numbers converging to 1 such that $1 < x_{n+1} < x_n < R$, for all $n \ge 1$. Moreover, given an index $n \ge 1$, a function $f_n \in V$ can be found with $f_n(x_n) = 0$. Since V is compact, we may assume without loss of generality that $f_n \to f \in V$ in $\mathcal{H}(\mathbb{D}_R)$. Consider a sequence $\{g_n(z)\}_{n\ge1}$ of functions defined by

$$g_n(z) = \frac{1}{1 - x_n z} + \frac{x_n\bar{z}}{1 - x_n\bar{z}}.$$

Clearly, $g_n \in \mathcal{H}(\mathbb{D}_{1/x_1})$. Moreover, it is easy to show that

$$g_n(z) \longrightarrow e(z) = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1-|z|^2}{|1-z|^2} \in \mathcal{H}\big(\mathbb{D}_{1/x_1}\big) \quad \text{as} \quad n \longrightarrow \infty,$$

and therefore, it follows from Lemma 2.1 that $f_n * g_n \to f * e = f$ in $\mathcal{H}\big(\mathbb{D}_{R/x_1}\big)$. In particular,

$$f_n(x_n) = (f_n * g_n)(1) \longrightarrow f(1) \quad \text{as} \quad n \longrightarrow \infty,$$

whence $f(1) = 0$, contrary to the hypotheses of the lemma. □

Lemma 2.4 *Let $V \subset U \subset \mathcal{H}_1$, and for each $\lambda \in \Lambda$ with $\lambda := g$, assume that $a_1(g) \in \lambda(V)$. Then the following assertions are equivalent*:

(a) $\lambda(U) = \lambda(V)$ *for all* $\lambda \in \Lambda$;
(b) $0 \notin \lambda(V) \Longrightarrow 0 \notin \lambda(U)$ *for all* $\lambda \in \Lambda$;
(c) $U^\top = V^\top$.

P r o o f. As the implication (a) $\Rightarrow$ (b) is trivially true, we first show that (b) $\Rightarrow$ (c). Let $g \in V^\top$ and $\lambda := g$. Then, by the definition of $\lambda(V)$, the assumption (b) shows that $\lambda(f) = (f * g)(1) \neq 0$ for all $f \in V$, that is, $0 \notin \lambda(V)$ and $0 \notin \lambda(U)$. Now for each $f \in U$ yields $(g * f)(1) \neq 0$, so that $g \in U^\top$. The other inclusion $U^\top \subset V^\top$ being obvious yields $U^\top = V^\top$.

Finally, we show that (c) $\Rightarrow$ (a). Fix a functional $\lambda \in \Lambda$ such that $\lambda := g$. It suffices to prove that $\lambda(U) \subset \lambda(V)$. Assume that $w \in \mathbb{C} \setminus \lambda(V)$ and so $w \neq a_1(g)$. For $f \in V$,

$$\lambda(f) - w = \{[g(z) - wz] * f(z)\}(1) \neq 0,$$

and therefore

$$\frac{g(z) - wz}{a_1(g) - w} \in V^\top = U^\top.$$

Equivalently, $w \notin \lambda(U)$ and thus, $\lambda(U) \subset \lambda(V)$. □

Remark 2.5 The condition (b) in Lemma 2.4 for $U = V'$ coincides with the condition (ii′) from [6, p. 64]. We next show that for a compact class V the additional assumption $a_1(g) \in \lambda(V)$ may be omitted.

Lemma 2.6 *Let $V \subset \mathcal{H}_1$ be compact, and $(V')^\top = V^\top$. If $\lambda \in \Lambda$ with $\lambda := g$, then $a_1(g) \in \lambda(V)$.*

P r o o f. This lemma is proved by contradiction. Assume that $a_1(g) \notin \lambda(V)$ for some $\lambda \in \Lambda$ with $\lambda := g$. Obviously, the set $\lambda(V) \subset \mathbb{C}$ is compact, and therefore an $\varepsilon > 0$ can be found such that $\mathbb{D}(a_1(g), \varepsilon) \cap \lambda(V) = \emptyset$. Now, if $w \in \mathbb{D}(a_1(g), \varepsilon)$, $w \neq a_1(g)$, then, by reasoning as in the proof of Lemma 2.4, we deduce that $w \notin \lambda(V')$. Thus, for all $f \in V$ and $x \in \overline{\mathbb{D}} \setminus \{0\}$,

$$w \neq \lambda(P_x f) = \frac{(g * f)(x)}{x}$$

and

$$a_1(g) = \lim_{x \to 0} \frac{(g * f)(x)}{x}.$$

The latter is an isolated point of the image of $\overline{\mathbb{D}}$ mapped by the function $F(z) = (g * f)(z)/z$ for each fixed $f \in V$. In fact, since F admits a continuous extension to the whole disk $\overline{\mathbb{D}}$, it must be constant. Thus,

$$\lambda(f) = (g * f)(1) = a_1(g),$$

contrary to the original assumption. □

Furthermore, note that $V^\top$ is complete if and only if $V^\top = (V')^\top$. Indeed, if $V^\top$ is complete, then $P_x g \in V^\top$ whenever $g \in V^\top$ and $x \in \overline{\mathbb{D}}$. This observation shows that

$$(P_x g * f)(1) = (g * P_x f)(1) \neq 0 \quad \text{for each} \quad f \in V, \tag{2.4}$$

whence $g \in (V')^\top$. Clearly, $V \subset V'$ and from (2.4) it follows that $V^\top = (V')^\top$. Conversely, if the latter equality holds, then, in view of the condition (2.4), we conclude that $g \in V^\top$ implies $P_x g \in V^\top$ for all $x \in \overline{\mathbb{D}}$.

3 Proofs of the main results and consequences

P r o o f o f T h e o r e m 1.2. By virtue of Lemmas 2.4 and 2.6, in order to prove the first assertion of Theorem 1.2, it suffices to verify the inclusion $V^\top \subset (V^{**})^\top$. Let $g \in V^\top = (V')^\top$. Then $g \in \mathcal{H}(\mathbb{D}_R)$ for some $R > 1$, and $(g * f)(1) \neq 0$ for each function $f \in V'$. By the compactness of V' in $\mathcal{H}$, it is evident from Lemmas 2.2 and 2.3 that for some $\sigma, 1 < \sigma < R$, the inequality $(g * f)(\sigma) \neq 0$ holds for all $f \in V'$. Next, let $h = P_\sigma g$, so that $h \in \overline{\mathcal{H}} \cap \mathcal{H}_1$ and $(h * f)(1) \neq 0$ for each $f \in V'$. Thus $h \in (V')^\top$ and it is easily verified that $h \in V^*$. Therefore, for an arbitrary $k \in V^{**}$,

$$(g * k)(1) = (P_{1/\sigma} h * k)(1) = \sigma(h * k)(1/\sigma) \neq 0,$$

whence $g \in (V^{**})^\top$.

Now, we proceed to the proof of the second part of Theorem 1.2. As shown above, $\lambda(V) = \lambda(V^{**})$ for all $\lambda \in \Lambda$ and from $f \in V^{**}$, it follows that $\lambda(f) \in \lambda(V)$.

Conversely, let $\lambda(f) \in \lambda(V)$ for $\lambda \in \Lambda$. Taking an arbitrary $g \in V^*$ and $z \in \check{\mathbb{D}}$, consider the functional $\lambda_z := P_z g$. It follows that

$$\lambda_z(f) = (P_z g * f)(1) = \frac{(g * f)(z)}{z} \neq 0 \quad \text{for all} \quad z \in \check{\mathbb{D}}$$

and thus, $f \in V^{**}$. □

P r o o f o f T h e o r e m 1.3. Let $h \in V^{**}$, $g \in V^{\top}$ and $\lambda := g$. Then, for each $f \in V$,

$$\lambda(f) = (g * f)(1) \neq 0, \quad \text{that is,} \quad 0 \notin \lambda(V).$$

By Theorem 1.2, $\lambda(h) \in \lambda(V)$ so that $0 \neq \lambda(h) = (g * h)(1)$ and therefore, $h \in (V^{\top})^{\perp}$.

On the other hand, if $h \in (V^{\top})^{\perp}$, $g \in V^*$ and $z \in \check{\mathbb{D}}$, then it is clear that $P_z g \in V^{\top}$. But then

$$(P_z g * h)(1) = \frac{(g * h)(z)}{z} \neq 0 \quad \text{for each} \quad g \in V^*, \quad \text{and all} \quad z \in \mathbb{D}.$$

Therefore, $h \in V^{**}$. □

Corollary 3.1 *Under the conditions of Theorem* 1.2*, then*

$$V^{**} = \bigcap_{\lambda \in \Lambda} \lambda^{-1}[\lambda(V)] = \bigcap_{\lambda \in \Lambda} (V + \ker \lambda),$$

where $\lambda^{-1}(A)$ denotes the inverse image of the set A with respect to λ. Here $A + B$ stands for the algebraic sum of the sets A and B, namely,

$$A + B = \{x + y : \, x \in A, y \in B\},$$

and $\ker \lambda$ *is the kernel of the functional λ defined by* $\ker \lambda = \{f \in \mathcal{H} : \, \lambda(f) = 0\}$.

P r o o f. According to Theorem 1.2, $f \in V^{**}$ is equivalent to the fact that $\lambda(f) \in \lambda(V)$ for each $\lambda \in \Lambda$, which, in turn, is equivalent to $f \in \lambda^{-1}[\lambda(V)]$ for each $\lambda \in \Lambda$ and therefore, the first equality is proved.

Now, if $f \in \lambda^{-1}[\lambda(V)]$ then there exists a function $g \in V$ satisfying $\lambda(g) = \lambda(f)$. Thus, for $h = f - g$, then $\lambda(h) = \lambda(f) - \lambda(g) = 0$ and so, $h \in \ker \lambda$. Therefore, $f = g + h \in V + \ker \lambda$. On the other hand, if $f = g + h$, where $g \in V$ and $h \in \ker \lambda$, then $\lambda(f) = \lambda(g) \in \lambda(V)$. □

Remark 3.2 If $V \subset \mathcal{H}_1$ is a compact dual class, then $V = \bigcap_{\lambda \in \Lambda} \lambda^{-1}[\lambda(V)]$. A similar representation is valid for an arbitrary compact convex set in a locally convex space X. However, in this case, Λ should be replaced by the space of all real-valued continuous linear functionals on X, see [9, p. 88].

Corollary 3.3 *If $U, V \subset \mathcal{H}_1$ are compact and $U^{\top}, V^{\top}$ are complete classes, then the following relations are equivalent*:

(a) $U^{**} = V^{**}$
(b) $U^{\top} = V^{\top}$
(c) $U^* = V^*$.

P r o o f. To begin with, assume that the relation (a) holds. Let $g \in U^{\top}$ with $\lambda := g$. It follows from Theorem 1.2 that

$$0 \notin \lambda(U) = \lambda(U^{**}) = \lambda(V^{**}) = \lambda(V),$$

and so, $g \in V^{\top}$. Thus $U^{\top} \subseteq V^{\top}$. The opposite inclusion is proved in a similar way. Thus, (a) $\Rightarrow$ (b) holds.

Next, we show the implication (b) ⇒ (c). Suppose that (b) is valid, and $g \in V^*$. Then, for each $f \in V$ and for all $z \in \check{\mathbb{D}}$,

$$0 \neq \frac{(g * f)(z)}{z} = (P_z g * f)(1)$$

so that $P_z g \in V^\top = U^\top$. Now the previous inclusion holds for an arbitrary $f \in U$, showing that $g \in U^*$, that is, $V^* \subseteq U^*$. By reversing the argument, it follows that (b) ⇒ (c).

The implication (c) ⇒ (a) is obvious, and this completes the proof. □

P r o o f o f T h e o r e m 1.4. Since $V^\top$ is complete, each $r \in (0,1)$ yields $P_r(V^\top) \subset V^\top$, and therefore

$$\bigcup_{0<r<1} P_r(V^\top) \subset V^\top.$$

Let $g \in V^\top$. Then $g \in \mathcal{H}(\mathbb{D}_R)$ for some $R > 1$, and $(g * f)(1) \neq 0$ for all $f \in V$.

By virtue of Lemmas 2.2 and 2.3, there exists a positive real number $\sigma \in (1, R)$ such that $(g * f)(\sigma) \neq 0$ for all $f \in V$. Setting $h = P_\sigma g$ (so that $h \in \overline{\mathcal{H}}$) yields

$$(h * f)(1) = (g * f)(\sigma)/\sigma \neq 0 \quad \text{for all} \quad f \in V,$$

and therefore $h \in V^\top$. Thus,

$$g = P_{1/\sigma} h \in \bigcup_{0<r<1} P_r\big(V^\top\big).$$

To prove the second equality, fix $g \in V^\top$ and $r \in (0,1)$. By the completeness of $V^\top$, we deduce that $(g * P_r f)(1) \neq 0$ for all $f \in V$, and therefore, $g \in (P_r V)^\top$.

Now, the inclusion $V^\top \subset (P_r V)^\top$ yields $P_r\big(V^\top\big) \subset \big[P_r(P_r V)^\top\big]$, and thus,

$$V^\top = \bigcup_{0<r<1} P_r\big(V^\top\big) \subset \bigcup_{0<r<1} P_r\big[(P_r V)^\top\big].$$

On the other hand, if g belongs to the right-hand side of the above inclusion, then $g \in P_r\big[(P_r V)^\top\big]$ for some $r \in (0,1)$ showing that $g = P_r h$, with $h \in (P_r V)^\top$. Clearly, $g \in \overline{\mathcal{H}} \cap \mathcal{H}_1$, and

$$(h * P_r f)(1) = (P_r h * f)(1) \neq 0 \quad \text{for all} \quad f \in V.$$

Consequently, $g \in V^\top$. □

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